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Some advances in Pure Mathematics made in the 19th century

By **Professor Dusa McDuff, FRS, CorrFRSE (Professor of Mathematics at Barnard College) and David Forfar, FFA, FRSE (Trustee of the James Clerk Maxwell Foundation)**

"Maxwell was not, and certainly never attempted to be, in the foremost rank of mathematicians. He preferred always to have before him a geometrical or physical representation of the problem in which he was engaged, and to take all his steps with the aid of this: afterwards, when necessary, translating them into symbols."

Obituary Tribute to Maxwell in 1879 by schoolfellow and scientific colleague, Professor P. G. Tait,[1] (1831–1902),

Introduction

During the 19th century, important advances were made in both physics and mathematics.

In physics, particularly in electricity and magnetism, the names of Faraday, Maxwell, Hertz and Marconi are well known to physicists and engineers.

In mathematics, the names of Gauss, Galois, Riemann and Noether are well known to mathematicians. Several of the advances in mathematics were later to prove crucial in 20th century physics.

Euclidean and Non-Euclidean Geometry

The first four postulates of Euclid date back more than 2,000 years and are part of early education in geometry. They are:

1) A straight line may be drawn between any two points,
2) Any straight line may be indefinitely extended,
3) A circle may be drawn with any given point as centre and any given radius,
4) All right angles are equal.

The fifth postulate of Euclid is that:

5) through any point not on a given line, there is only one line which is parallel to the given line (the parallel postulate).

The philosopher Immanuel Kant had thought (Fig.1) in the 18th century that all geometry was, a priori, Euclidean.

Figure 1: Emmanuel Kant (1724–1804), Wikipedia Commons

However, by the 1830s, it was becoming clear, through the work of the Russian mathematician Nikolai Lobachevsky and the Hungarian mathematician János Bolyai, that, despite all the attempts that had been made, Euclid's fifth postulate (the parallel postulate) could not be proved from the first four postulates of Euclid and was independent of them.

Figure 2: Nikolai Lobachevsky (1792–1856), Wikipedia Commons

Lobachevsky and Bolyai had discovered that, apart from the familiar geometry of Euclid, where all five of the above postulates held true, there was an alternative geometry where the first four postulates of Euclid held true but Euclid's parallel postulate did not. In this alternative geometry, the sum of the angles of a triangle was less than 180° and parallel lines diverged.

Figure 3: Janos Bolyai (1802–1860), Wikipedia Commons

1 Professor P. G. Tait (1831 – 1901), Professor of Physics at Edinburgh University; Maxwell's chief scientific correspondent.

For illustration of this alternative geometry, we may consider a very small creature – a bug – moving around on a two dimensional surface. We assume that the bug is purely confined to the surface and is not conscious of any dimension other than along the surface. The bug can find the shortest distance between any two points on the surface by means of a taut string running between the two points. These are the straight lines of the bug's geometry. Furthermore, the bug can draw circles of any radius on the surface (Euclid's third postulate) and also take length and area measurements.

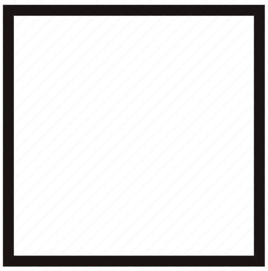

Figure 4: Flat Surface

We first consider the bug moving around on the flat surface of a square (Fig. 4). The bug chooses any three points on the surface, finding the shortest distance between any two by use of a taut string (the bug's straight lines). He finds that the sum of the internal angles of any triangle thus drawn on the surface is 180° and parallel lines remain parallel. He concludes that the geometry of a flat surface is the usual geometry of Euclid.

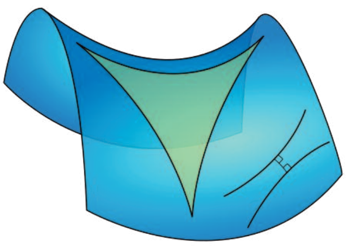

Figure 5: The surface of a saddle, courtesy tex.stackexchange.com

The bug next moves around on a second surface in the shape of a saddle (Fig. 5). Again, the bug chooses three points on the surface, finding the shortest distance between any two (by use of a taut string but keeping the string confined to the surface) and again draws a triangle. He finds that the sum of the angles of his triangle (Fig. 5) is less than 180° and lines drawn on the saddle, which were initially parallel, diverge. However, the bug finds that if his triangle is drawn ever smaller, then the sum of the angles of a triangle approaches ever nearer to 180°. The bug concludes that the geometry of this surface, in the large, is not Euclidean but, in the very small, is nearly Euclidean.

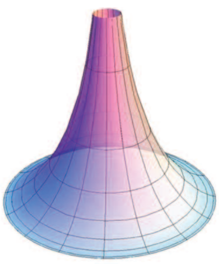

Figure 6: The surface of a pseudosphere, courtesy steemit.com

The bug next moves around on the surface of a pseudo-sphere (Fig. 6). As usual, the bug chooses three points on the surface, finds the shortest distance between any two of them and again draws a triangle on the surface. He finds that the sum of the angles of his triangle is less than 180°. However, the bug finds that if he draws the triangle ever smaller, the results become closer and closer to the results under Euclidean geometry. The bug concludes that the geometry of the pseudo-sphere surface is, in the large, not Euclidean although, in the very small, it is nearly Euclidean.

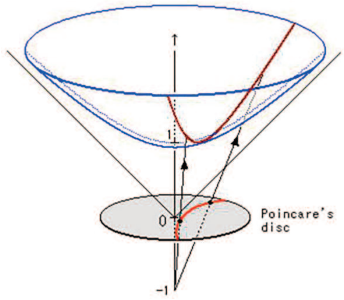

Figure 7: Hyperbolic surface and the projection into a Poincare disc, courtesy bjlkeng.github.io

The bug (which we shall now call the bowl-bug) next moves around on a bowl-like surface (Fig. 7) in the form of a hyperbola rotated round its axis. We imagine another bug (which we shall call, the disc-bug) moving round on the flat disc (shown in grey) in such a way that (a) the bottom point labelled, -1 (b) the bowl-bug and (c) the disc-bug are all inline – see the two projection lines with arrows in Fig. 7. When the bowl-bug moves on the dark red lines (the bowl-bugs straight lines), the disc-bug moves on the bright red line. Both bugs find that the angles of a triangle add up to less than 180°. Both bugs find that their geometry is not Euclidean except that, for very small figures, the geometry tends, in the limit, towards being Euclidean.

The flat disc is known as the Poincaré disc and is an illustration, on a flat plane, of the geometric figures arising in the non-Euclidean geometry ('hyperbolic geometry') of Lobachevsky and Bolyai.

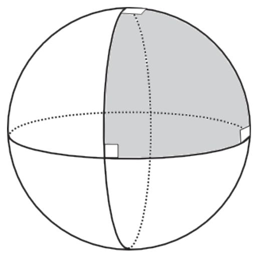

Figure 8: Triangle drawn on a sphere where the three angles add up to 270°, courtesy homepages.ucl.ac.uk

The bug next moves around on the surface a sphere (Fig. 8). Again, the bug chooses three points on the surface, finds the shortest distance between any two (by use of a taut string but always confining the string to the surface) and draws a triangle. He finds that the sum of the angles of his triangle is more than 180° and lines which, at the equator, start parallel (i.e. lines of longitude) all meet up at the North pole. For example, the bug finds that the angles of the grey triangle (Fig. 8) add up to 270°. But the bug finds that the smaller he draws the triangle, the more the surface geometry becomes closer to Euclidean. The bug concludes that the geometry of the surface of the sphere is, in the large, non-Euclidean although, in the very small, it is nearly Euclidean.

These examples show that curved, smooth surfaces exhibit a geometry that is, in the large, non-Euclidean, but, in the small, tends towards being Euclidean. This is particularly true if the points on a curved surface are projected onto a flat plane. We are familiar with Mercator's projection of the Earth distorting distances.

Figure 9: Carl Friedrich Gauss (1777–1855), Wikipedia Commons

The analysis of non-Euclidean geometries was considered by Gauss and Riemann. Gauss had considered the geometry of two dimensional curved surfaces, as in the examples above. Riemann extended Gauss' work to spaces of three, four or any number of dimensions which have, in the large, a non-Euclidean geometry but in the very small, a Euclidean geometry. Such smooth surfaces are called, in mathematics, 'manifolds'.

Figure 10: Bernhard Riemann (1926–1866), Wikipedia Commons

In the 20th century, the application of non-Euclidean geometry lead to significant advances in physics. For example, in order to take account of gravity, Einstein required, for his description of space, a four dimensional non-Euclidean geometry with the normal three dimensions (x,y,z) of space plus the addition of the time dimension, *t*. For a heavy body, the attractive force of gravity ensures that the trajectories of particles in 'free fall' (i.e. the straight lines in this geometry) converge towards the centre of the heavy body thus gravity exhibits a non-Euclidean character.

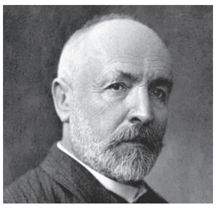

Figure 11: Georg Cantor, (1845–1918), Wikipedia Commons

Cantor, orders of infinity and the 'continuum hypothesis'

In the 1890s, Georg Cantor showed that an infinite set can be put into one-to-one correspondence with a second infinite set which is only a subset of the first set!

For example, the infinite set {1,2,3,4,...} can be put into one-to-one correspondence with the infinite set {2, 4, 6, 8,...}, which is only a subset of the first set. The one-to-one correspondence is formed by the pairing of *n* in the first set with *2n* in the second set.

The first infinite set may seem, at first glance, to be twice as large as the second infinite set. However, the existence of the one-to-one correspondence between the two sets makes, in Cantor's sense, these two infinite sets the same 'size' despite one being a subset of the other! Infinite sets behave rather differently from finite sets!

In 1892, Cantor showed that, in his sense, the infinite set consisting of all the 'real' numbers (i.e. all the integers and all the numbers expressed in finite or infinite decimals) was 'bigger' than the infinite set consisting of the 'positive integers' {1, 2, 3, 4,...} because there is no way to make each decimal number correspond to a different integer – there are just not enough integers to go round.

Cantor called the infinite set consisting of all the positive integers {1, 2, 3, 4...}, 'countable' whereas the infinite set of all the real numbers, being a larger infinity, was called 'uncountable'.

The concept that one infinite set could be 'bigger' (in Cantor's sense) than another infinite set was very radical and took quite a while to be accepted although now this idea is well-established in the mathematical canon.

However, Cantor was not able to prove what is called the 'continuum hypothesis' namely whether, or not, there is an infinite set whose size (in the Cantor sense) lies intermediate between the set of all the integers and the much bigger set of all the real numbers.

It was only in the 20^{th} century that the continuum hypothesis was found to lie outside the axioms of arithmetic and therefore could not be settled one way or the other within these axioms.

Different types of number

Mathematicians have always been interested in understanding numbers. Originally these arose as the lengths of lines in geometric figures. Thus any such quantity as the square root of two (which is the length of the diagonal of a unit square) is a number. In the 19^{th} century, the effort to clarify the different kinds of number led to many profound new ideas in mathematics. We begin by explaining the different types of number.

The 'integers' are the set of whole numbers {...-2, -1, 0, 1, 2, 3, 4...} and the 'rational' numbers (such as 2/3) are those formed by dividing one integer by another (rational being used in the sense of a ratio). Such numbers can all be written as decimals that are either finite or recurring, for example, 7/8=0.875 or 2/9=0.222222... or 15/273=0.054945054945... All the other real numbers are called 'irrational'. Their decimal expressions are non-recurring. For example, $\sqrt{2}$ = 1.4142135623... or π=3.1415926535...

→

The simplest irrational numbers are the 'radical' numbers. These are made from square roots and cube roots (or more generally, any higher root) of integers by using the usual arithmetic operations of addition, subtraction, multiplication and division. For example, the two solutions of the quadratic $ax^2+bx+c=0$ can be written as $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

The algebraic' numbers are the set of all the solutions to polynomial equations with rational coefficients.

The 'real' numbers are all the possible decimal numbers, whether they are integers or require their decimal representation to have a finite, recurring or non-recurring number of decimals. Most of the real numbers are not algebraic.

The 'complex' numbers in mathematics are those of the form ($a+bi$) where a and b are real numbers and i is the square root of -1. Complex numbers cannot be used for measuring lengths (such as in Euclidean geometry) so i is often said to be an imaginary number. However, the rules of arithmetic can be extended to include them and it turns out that they are very useful mathematical tools with many important applications in physics and engineering.

Construction with straightedge and compass

Some of the problems solved for the first time in the 19th century concerned problems regarding the different geometrical figures that could be drawn accurately with a ruler (in the sense of a straightedge without any markings) and a compass. In mathematical terms, such figures are called 'constructable'.

The ancient Greeks knew how to construct, using only ruler and compass, a triangle (a 3-gon), a square (a 4-gon) and a pentagon (a 5-gon), all with equal sides, and to bisect a general angle. They also knew how to construct an equal-sided polygon with double the number of sides of a given equal-sided polygon.

But the ancient Greeks had found that they could not construct, using only a straightedge and a compass, an equal-sided polygon having an arbitrary number of sides or trisect a general angle or duplicate a cube (i.e. given a cube of unit volume, the aim is to construct the side of a cube which has a volume of two units) or square a circle (i.e. given a circle, to find the side of a square of the same area). Was this just lack of effort, or was there a deep reason for this?

Little progress on these problems was made in the more than a thousand years between ancient times and the start of the 19th century.

It was not until 1796, that at the age of 19, the mathematician Carl Friedrich Gauss (mentioned above) made inroads into these problems. Gauss was one of the great mathematicians of the first half of the 19th century. He showed that an equal-sided polygon with n-sides could be constructed provided the number n was the product any number of distinct prime numbers of the form $(2^{2^n}+1)$ – being 3, 5 and 17 when n=0, 1 or 2 – multiplied by a power of 2.

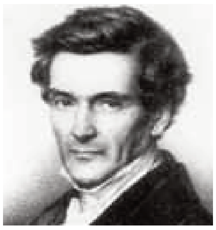

Figure 12:
Pierre Wanzel (1814–1848), Courtesy MacTutor

In 1837, Pierre Wantzel proved that it was impossible, with only straightedge and compass to (a) trisect a general angle and (b) to duplicate the cube.

Furthermore, the 19th century also provided the proof that it was not possible, by ruler and straightedge, to square the circle. If that had been possible, it would have been possible to construct the number $\sqrt{\pi}$. But Lindemann had proved in 1882 that the number π was transcendental (see later). It followed that as the number $\sqrt{\pi}$ was also transcendental. It was known that transcendental numbers were not constructible with only ruler and a compass.

We do not go into the proofs but each involved new mathematical ideas that were needed to understand the relevant kind of number.

Polynomials with integer coefficients

It was known by the 19th century, that all polynomial equations have solutions (which might be real numbers or complex numbers) and the number of solutions[2] was equal to the degree of the polynomial.

The great importance of this theorem in mathematics is shown by the fact that this theorem is known as the 'fundamental theorem of algebra'.

Quadratic equations

It was known since Babylonian times, that the two solutions of the quadratic equation with integer coefficients, namely $ax^2+bx+c=0$ are:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

2 Including the number of repeated solutions.

These two solutions involve purely radical expressions (see above) in the coefficients {a,b,c}.

The two solutions may require, for their expression, the use of complex numbers. For example, a complex number is required whenever b^2 is less than 4ac so the expression (b^2-4ac) in the coefficients is negative. In order to express the square root of a negative number, the number *i* is required. For example, the complex number $3+\sqrt{-4}=3+2i$

The formula for the cubic and quadric polynomial

It was known, from the 16th century onwards, that formulae, similar to the quadratic formula, although more complicated, existed for the solutions of all cubic and quartic polynomial equations, the formulae involving only radical expressions in the polynomial's integer coefficients.

However, the attempts to find a similar formula for the general quintic, and not just for some special quintics, did not meet with success. It was suspected that the attempt to find a formula, in terms of radical expressions, for all quintics might not be possible.

Nevertheless, the quest was pursued to find formulae, like that for the quadratic, which gave general solutions of quintic, and higher degree equations (in terms of their integer coefficients).

The reasons why this quest was unsuccessful remained a mystery for the 300 years between 16th and 19th century.

Figure 13: Neils Abel (1802–1829), Wikipedia Commons

Neils Abel

In about 1820, the Norwegian mathematician Neils Abel proved, via his ingenuity with algebra, that that the general quintic equation could not be solved by a general formula involving only radical expressions in the integer coefficients of the quintic. However, he did not develop the concepts which proved to be fundamental to answering the question; that was done by Evariste Galois.

Figure 14: Evariste Galois (1811–1832), Wikipedia Commons

Evariste Galois and Galois theory

In 1830s, Evariste Galois, while still in his teens, was able to develop the conditions that had to be satisfied for a general quintic or higher degree polynomial to be solvable using radical expressions in the integer coefficients, thereby solving a problem that had stood for 300 years.

An example of a quintic equation whose five roots exist but cannot be expressed in terms of radical expressions is $(x^5 - x + 1) = 0$.

Galois had proved that the set of all radical numbers was smaller than the set of algebraic numbers (i.e. the set of all solutions to polynomial equations with integer coefficients).

We do not go into the details as they are somewhat technical, suffice it to say that, in presenting his solution, Galois needed to originate the concept of a 'group'. This encodes the subtle symmetry properties of the set of all solutions of this equation. (Note that the word 'group' has a purely technical meaning in mathematics, quite different from its usual meaning).

The group concept has proved enormously fruitful in mathematics and is now central to many areas of physics (see Section on symmetry).

Galois had laid the foundations for a theory that is named after him, namely 'Galois Theory'. Galois Theory has played an important role in some of the most celebrated mathematical discoveries of the 20th century. These have included Andrew Wiles' 1994 proof of Fermat's Last Theorem[3].

Although Galois accomplished his investigations when he was just 19, he is now recognised as being one of the great mathematicians of the 19th century. He died in 1832 at age 20 from wounds suffered in a duel. His work was not published until 1846 by Liouville[4] and was not included in a textbook until the 1870s[5].

3 Fermat's famous Last Theorem is that $x^n+y^n=z^n$ has no integral solutions for (x,y,z) if n is a positive integer, other than n=1 and n=2.

4 Joseph Louiville, 1846, *'Oevres mathematique d'Evariste Galois'.* T. XI, 381-444

5 Camille Jordan, 1870, *'Traité des Substitutions et les Equations Algébrique'* (Wentworth Press)

Transcendental numbers

'Transcendental' numbers are numbers that do not satisfy any polynomial equation with integer coefficients, no matter what the degree of the equation. The number π is the best such example. Although only a few specific transcendental numbers are known, most numbers are transcendental. Indeed, the numbers that are the solutions to polynomial equations with integer coefficients (i.e. the set of all algebraic numbers) form a countable set, while the set of all transcendental forms an uncountable set and so is 'bigger' (in Cantor's sense) than any countable set.

Figure 15: Joseph Liouville (1809–1882), Wikipedia Commons

In 1851, Joseph Liouville was the first to express a transcendental number as an infinite number of non-recurring decimals. The number was: 0.1100010000000000000000010000... where there is a 1 in place n! (factorial n) and a zero otherwise. The 1s follow as 1! = 1, 2! = 2, 3! = 6, 4! = 24.

Figure 16: Charles Hermite (1822–1901), Wikipedia Commons

In 1873, Charles Hermite proved that the number which we now call 'e' (the base of Napierian logarithms) was transcendental.

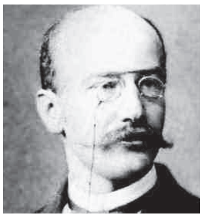

Figure 17: Ferdinand Lindemann (1852–1939), Wikipedia Commons

In 1882, Frederick Lindemann proved that the number π was also transcendental. A consequence of his proof was that it was not possible to 'square the circle' (i.e. to construct, with ruler and compass, a square equal in area to a given circle).

Symmetry and conservation laws

The word 'symmetry' has a wider meaning in mathematics and physics than just requiring that the visual appearance of the two sides of a picture be symmetrical.

In mathematics, symmetry refers to the result of certain actions on an object that change some of its features while leaving others unchanged.

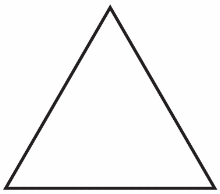

Figure 18: Symmetry of an Equilateral Triangle

For example, an equilateral triangle (Fig. 18) may be left alone (action 1) or may be turned by 120º (action 2) or by 240º (action 3) or flipped over (action 4). These actions do not affect the overall 'look' of the triangle but they move its vertices.

The actions can be combined to make a mathematical group (a symmetry group).

In physics, it may be found that the result of an experiment is independent of the place or the time where the experiment was carried out i.e. the experiment exhibits place-symmetry and time-symmetry. Furthermore, the experimental results may not depend on the direction in which the apparatus happened to be facing at the time i.e. the experiment exhibits rotation-symmetry.

Figure 19: Emmy Noether (1882–1935) Courtesy es.paperblog.com

In the early 20th century, the algebraist Emmy Noether proved a rather remarkable result, namely that any (continuous and smooth) symmetry implies a conservation law (Noether's Theorem). Emmy Noether was described by Albert Einstein as one of the great mathematicians of his era.

The first symmetry above implies the conservation of momentum, the second, the conservation of energy and the third, the conservation of angular momentum.

The notions of symmetry and a group of symmetries has now become fundamental in physics. For example, the notion of symmetry is central to the theory of elementary particles.

It is thought that Maxwell, had he been alive at the time, would have much appreciated the connection between symmetry and conservation laws as is implied by Noether's Theorem.

This story shows how ideas (such as that of a group) that first arose in a purely mathematical context can have profound implications for our general understanding of the physical world.

James Clerk Maxwell Foundation, 14 India Street, Edinburgh EH3 6EZ.
The birthplace in 1831 of James Clerk Maxwell.

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